ON BOUNDARY CORRESPONDENCE IN CONFORMAL MAPPING(Topics in Univalent Functions and Its Applications)

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ON BOUNDARY CORRESPONDENCE IN CONFORMAL MAPPING*

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1. INTRODUCTION

The prime end theory gives a complete but abstract description of the relation between topological properties of a domain and the contiuity behaviour of the mapping function (see [1], [2]). It is still necessary to study the boundary correspondence of conformal mappings in general sence. Some results about it have been obtained, for instance, by M. Essen [3] and Chr. Pommerenke [4].

The aim of this paper is to study how the local geometric structure of a domain at its boundary points determines analytic properties of the mapping function. For this reason we give a classification of the boundary points of a domain.

Through this paper, except for few cases, G is always a simply connected domain with at least two boundary points, $\infty \notin \partial G$, and f is a conformal mapping of D onto G. For $\zeta \in \partial D$ the expression $f(\zeta)=w$ means f has the angular limit w at ζ. Let

$$f^{-1}(w)=\{\zeta\in\partial D:\ f(\zeta)=w\} \tag{1.1}$$

$$A(w,f)=\{\zeta\in\partial D:\ w\in C(f,\zeta)\} \tag{1.2}$$

where $C(f,\zeta)$ is the cluster set of f at ζ.

For $w\in\partial G$ let $N_\varepsilon(w)$ denote the ε-neighbourhood of w. We divide the connected components of $N_\varepsilon(w)\cap G$ into two parts: the set $P_0(w,\varepsilon)$ of all components with w as a boundary point and the set $P_1(w,\varepsilon)$ of the other components. Let

$$\begin{aligned} d(\varepsilon,w) &= \inf\{\operatorname{dist}(w,V_W(\varepsilon)):\ V_W(\varepsilon)\in P_1(w,\varepsilon)\};\\ d(\varepsilon,w) &= \varepsilon, \quad \text{if } P_1(w,\varepsilon)=\emptyset. \end{aligned} \tag{1.3}$$

If there exists $\varepsilon_0>0$ with $d(\varepsilon_0,w)=0$ then w is called a complicated boundary point of G, otherwise w is a simple one.

Note that w is a simple boundary point of G if and only if G is locally connected at w. The classification applies to open sets.

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In this paper it has been proved that

(i) w is a simple boundary point of G if and only if

$$A(w,f)=f^{-1}(w);$$

(ii) if G is bounded then any analytic univalent function in G has no Koebe arcs if and only if every boundary point of G is simple;

(iii) a compact set A in complex plane is locally connrcted if and only if every boundaty point of its complement is simple and the number of all connected components of A is finite;

(iv) let $|E|$ denots the number (integer or ∞) of all points of the point set E. Writing $p_0(\varepsilon)=|P_0(w,\varepsilon)|$, we have

If w is a simple boundary point of G then $p_0(\varepsilon)$ is increasing when ε is decreasing and

$$|f^{-1}(w)|=\lim_{\varepsilon\to 0} p_0(\varepsilon)=|A(w,f)|;$$

If w is a complicted boundary point of G then

$$|f^{-1}(w)|=\varliminf_{\varepsilon\to 0} p_0(\varepsilon)\leqslant|A(w,f)|$$

with strict inequality if $|f^{-1}(w)|<\infty$.

Our discussion begins with a research into accessible boundary points. $w\in\partial G$ is accessible in G if and only if there exists a sequence $V_w(\varepsilon_n)\in P_0(w,\varepsilon_n)$, $n=1,2\cdots$, with $V_w(\varepsilon_1)\supset V_w(\varepsilon_2)\supset\cdots$ and $\varepsilon_n\to\infty$. Such a sequence is called a regular component sequence at w. Two regular sequences $\{V_w(\varepsilon_n)\}$ and $\{v_w'(\varepsilon_n')\}$ are called to be not equivalent each other if there exists $N>0$ such that $V_w(\varepsilon_n)\cap V_w'(\varepsilon_n')=\emptyset$ for $n>N$. There exists a one-to-one correspondence between $f^{-1}(w)$ and $M(w)$ where $M(w)$ is a complete set of inequivalent regular sequences at w.

Whether there exists a domain which has no simple boundary poijnts is an open problem. A domain that all of its simple boundary points is a set of zero linear measure has been comstructed.

2. ON ACCESSIBLE BOUNDARY POINTS

The results to be developed in this section are, with modifications, due to X. Yang[5].

We need the following lemma which is easy to provs.

Lemma 2.1 Let γ_n $(n=1,2,\cdots)$ be a sequence of Jordan arcs which do not intersect each other except for a common end of γ_n and γ_{n+1}, $n=1,2,\cdots$. If $\gamma_n\to w$ $(n\to\infty)$ and $w\notin\gamma_n$ for any n, then

$$\gamma=\{w\}\cup\bigcup_{n=1}^{\infty}\gamma_n \tag{2.1}$$

is a Jordan arc with w as an end.

Theorem 2.1 For $w\notin\partial G$ the following conditions are equivalent:

(i) There exists a regular component sequence at w;

(ii) w is an accessible boundary point of G;

(iii) There exists $\zeta \in \partial D$ with $f(\zeta) = w$.

Proof (i)$\Rightarrow$(ii): Let $\{V_w(\varepsilon_n)\}$ be a regular component sequence at w. We may assume $V_w(\varepsilon_n) \setminus V_w(\varepsilon_{n+1}) \neq \emptyset$, $n = 1, 2, \cdots$. Taking

$$w_n \in V_w(\varepsilon_n) \setminus V_w(\varepsilon_{n+1}) \tag{2.2}$$

for $n = 1, 2$, let $\Gamma_1 \subset V_w(\varepsilon_1)$ be a Jordan are with ends w_1 and w_2. We may suppose that $\varepsilon_3 < \operatorname{dist}(w, \Gamma_1)$. Then we take w_3 as (2.2) with $n = 3$ and a Jordan arc $\Gamma_2 \subset V_w(\varepsilon_2)$ joining w_2 and w_3. Let γ_1 denote the subarc of Γ_1 between w_1 and the first intersection w_1' with Γ_2.

Repeating the preceding process, we may suppose $\varepsilon_4 < \operatorname{dist}(w, \Gamma_2)$ and take w_4 as (2.2) with $n = 4$ and then a Jordan arc $\Gamma_3 \subset V_w(\varepsilon_3)$ joining w_3 and w_4. Let γ_2 denote the subarc of Γ_2 betweem w_1' and the first intersection w_2' with Γ_3.

Continuing this program infinite times we obtain a sequence $\{\gamma_n\}$ and then a Jordan arc γ defined by (2.1) which lies in G except for the end w by Lemma 2.1.

The implication (ii)$\Rightarrow$ (iii) is obvious (see [2]).

(iii)$\Rightarrow$ (i): The condition (iii) implies $\lim_{r \to 1} f(r\zeta) = w$. Therefore, for any strictly decreasing sequence $\{\varepsilon_n\}$ tending to zero there exists a corresponding positive sequence $\{r_n\}$ tending to 1 such that

$$\{ f(r\zeta) : r_n \leq r < 1 \} \subset N_{\varepsilon_n}(w) \cap G$$

Let $V_w(\varepsilon_n)$ be the component of $N_{\varepsilon_n}(w) \cap G$ which contains $\{f(r\zeta) : r_n \leq r < 1\}$. Then $V_w(\varepsilon_n) \in P_0(w, \varepsilon_n)$ and $\{V_w(\varepsilon_n)\}$ is a regular component sequence at w.

Lemma 2.2 for $w \in \partial G$ the following conditions are equivalent:

(i) There exist two inequivalent regular component sequences at w;

(ii) There exist two distinct points ζ_1 and ζ_2 on ∂D such that $f(\zeta_1) = f(\zeta_2) = w$.

Proof (i)$\Rightarrow$ (ii): Let $\{V_w(\varepsilon_n)\}$ and $\{V_w'(\varepsilon_n)\}$ be two inequivalent component sequences at w. We may assume that $V_w(\varepsilon_n) \cap V_w'(\varepsilon_n) = \emptyset$, $n = 1, 2, \cdots$. From the proof of Theorem 2.1 there exist two Jordan arcs γ_1 and γ_2 that end at w and lie otherwise in $V_w(\varepsilon_1)$ and $V_w'(\varepsilon_1)$ respectively. Therefore, $f^{-1}(\gamma_1 \setminus \{w\})$ and $f^{-1}(\gamma_2 \setminus \{w\})$ are two asymptotic paths of f, say, end at ζ_1 and ζ_2 on ∂D respectively, and then $f(\zeta_1) = f(\zeta_2) = w$ (see [2]). We have to prove $\zeta_1 \neq \zeta_2$.

Suppose $\zeta_1 = \zeta_2 = \zeta$. Let w_j be the other end of γ_j, $j = 1, 2$. We join w_1 and w_2 by a Jordan arc Γ in G such that $\gamma_1 \cup \Gamma \cup \gamma_2$ is a Jordan curve whose inner domain is denoted by G_1. Let $D_1 = f^{-1}(G_1)$, then $f_1 = f|_{D_1}$ is a conformal mapping of D_1 onto G_1 and can be extended to a homeomorphism of $\overline{D_1}$ onto $\overline{G_1}$. There exist

$$z_n \in f^{-1}(\gamma_1 \setminus \{w\}), \quad z_n' \in f^{-1}(\gamma_2 \setminus \{w\}) \qquad \text{with } z_n \to \zeta,\ z_n' \to \zeta \ (n \to \infty)$$

such that $[z_n, z_n'] \subset \overline{D_1}$, $n = 1, 2 \cdots$. Let $w_n = f(z_n)$, $w_n' = f(z_n')$, then $w_n \in V_w(\varepsilon_1)$,

$w_n' \in V_w'(\varepsilon_1)$. It is clear that $\Gamma_n = f([z_n, z_n'])$ is a Jordan are joining w_n and w_n' in G with $\Gamma_n \to w$ $(n \to \infty)$, and so $w_n' \in V_w(\varepsilon_1)$ for large n, contrary to $V_w(\varepsilon_1) \cap V_w'(\varepsilon_1) = \emptyset$.

(ii)$\Rightarrow$ (i): Let $\zeta_1, \zeta_2 \in \partial D$, $\zeta_1 \neq \zeta_2$ with $f(\zeta_1) = f(\zeta_2) = w$. By using the method in the proof of Theorem 2.1, from the Jordan arcs $f(r\zeta_1)$ and $f(r\zeta_2)$ $(0 < r < 1)$, we obtain two regular component sequences at w, say, $\{V_w(\varepsilon_n)\}$ and $\{V_w'(\varepsilon_n)\}$ respectively. We have to prove that they are not equivalent.

If they are equivalent then $V_w(\varepsilon_n) = V_w'(\varepsilon_n)$, $n = 1, 2, \cdots$. Let $\{r_n\}$ be a positive sequence decided by $\{\varepsilon_n\}$ as that of the proof of Theorem 2.1, and then let $w_n = f(r_n\zeta_1)$, $w_n' = f(r_n\zeta_2)$. There exists a Jordan arc γ_n joining w_n and w_n' in $V_w(\varepsilon_n)$. Let $\Gamma_n = f^{-1}(\gamma_n)$, we have

$$\lim_{n \to \infty} \operatorname{diam} \Gamma_n \geq |\zeta_1 - \zeta_2| > 0 \quad \text{and} \quad f(\Gamma_n) \to w \ (n \to \infty)$$

It is impossible because f has no Koebe arcs (see [2]).

It follows that

Theorem 2.2 There exists a one-to-one correspondence between $M(w)$ and $f^{-1}(w)$.

3. THE CHARACTERISTIC OF TWO KINDS OF BOUNDARY POINTS

The following theorem is basic in this paper.

Theorem 3.1 $w \in \partial D$ is simple if and only if $A(w, f) = f^{-1}(w)$.

Proof (a) Let w be a simple boundary point of G. Suppose that $A(w, f) \setminus f^{-1}(w) \neq \emptyset$ then f has no asymptotic paths at any point of the set $A(w, f) \setminus f^{-1}(w)$.

Let $\zeta \in A(w, f) \setminus f^{-1}(w)$. Since f has angular limits almost everywhere on ∂D and $f^{-1}(w)$ is a set of zero capacity (see [2]), hence there exist two point sequences ζ_n' and ζ_n'' of ∂D which tend to ζ from two sides of ζ asymptotically such that f has angular limits different from w at ζ_n' and ζ_n''. Let S_n denote the arched domain bounded by $[\zeta_n', \zeta_n'']$ and $\widehat{\zeta_n'\zeta_n''}$ which contains ζ. We have $S_{n+1} \subset S_n$ and $d_n = \operatorname{dist}(w, f([\zeta_n', \zeta_n''])) > 0$, We take a positive decreasing sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ $(n \to \infty)$ such that

(i) $\varepsilon_{n+1} < d(\varepsilon_n, w)$, $n = 1, 2, \cdots$;

(ii) $\varepsilon_n < d_n$, $n = 1, 2, \cdots$;

From (i) we obtain

$$\{N_{\varepsilon_{n+1}}(w) \cap G\} \subset \bigcup V_w(\varepsilon_n) \tag{3.1}$$

where the union is for all $V_w(\varepsilon_n) \in P_0(w, \varepsilon_n)$.

From (ii) we have

$$\{N_{\varepsilon_{n+1}}(w) \cap f(S_{n+1})\} \subset \bigcup V_w(\varepsilon_n) \tag{3.2}$$

where the union is for all $V_w(\varepsilon_n) \in P_0(w, \varepsilon_n)$ with $V_w(\varepsilon_n) \in f(S_n)$.

Note that w is also a simple boundary point of each component of the last union. Then there exists a regular component sequence $\{V_w(\varepsilon_n)\}$ such that $V_w(\varepsilon_n)\subset f(S_n)$, that is $f^{-1}(V_w(\varepsilon_n))\subset S_n$, $n=1,2,\cdots$. And so we obtain a Jordan arc γ as we do in the proof of Theorem 2.1 such that $f^{-1}(\gamma)$ is an asymptotic path of f at ζ. This is a contradiction. It means $A(w,f)=f^{-1}(w)$.

(b) Suppose $A(w,f)=f^{-1}(w)$. If w is a complicated boundary point of G, we have $d(\varepsilon_0,w)=0$ for some $\varepsilon_0>0$. And then there exists a sequence $\{w_n\}$ in G with $w_n\in V_w'(\varepsilon_0)\in P_1(w,\varepsilon_0)$, $w_n\to w$ $(n\to\infty)$. Let $P_0'(w,\varepsilon_0)$ denote the set of all components in $P_0(w,\varepsilon_0)$ with w as an accessible boundary point. We have

$$z_n=f^{-1}(w_n)\in D_1=D\setminus\bigcup f^{-1}(V_w(\varepsilon_0))$$

where the union is for all $V_w(\varepsilon_0)\in P_0'(w,\varepsilon_0)$. Obviously, $f^{-1}(w)\bigcap\partial D_1=\phi$ and then every limit point of $\{z_n\}$ belongs to $A(w,f)\setminus f^{-1}(w)$, contrary to $A(w,f)=f^{-1}(w)$. So w is simple.

Corollary $w\in\partial G$ is complicated if and only if $A(w,f)\setminus f^{-1}(w)\neq\phi$.

It is clear that the numbers $|A(w,f)|$ and $|f^{-1}(w)|$ are independent of the choice of the mapping function f. They are called the multiplicity and the pseudo-multiplicity of the boundary point w, and denoted by m and p, respectively. Set $p_0(\varepsilon)=|P_0(w,\varepsilon)|$. We have

Theorem 3.2 Let $w\in\partial G$ with multiplicity m and pseudo-multiplisity p.

(i) If w is simple, then

$$p=\lim_{\varepsilon\to 0}p_0(\varepsilon)=m. \tag{3.3}$$

(ii) If w is complicated, then

$$p=\varliminf_{\varepsilon\to 0}p_0(\varepsilon)\leqslant m \tag{3.4}$$

with strict inequality if $p<\infty$.

Proof Let w is a simple boundary point of G, then $p_0(\varepsilon)\geqslant 1$ for any $\varepsilon>0$. Note that w is also simple for each component in $P_0(w,\varepsilon)$. This implies that $p_0(\varepsilon)$ increases when ε decreases. Let $p_0=\lim p_0(\varepsilon)$. We choose a positive decreasing sequence $\{\varepsilon_n\}$ with $\varepsilon_n\to 0$ $(n\to\infty)$ satisfying $\varepsilon_{n+1}<d(\varepsilon_n,w)$ $n=1,2\cdots$. Then we have (3.1).

If $p_0<\infty$ then there exists n_0 such that $p_0(\varepsilon_n)=p_0$ for $n\geqslant n_0$. So we obtain P_0 inequivalent regular component sequences at w and then (3.3) is true by Theorems 2.2 and 3.1.

If $p_0=\infty$ then for any given positive integer N we have $p_0(\varepsilon_n)\geqslant N$ when $n\geqslant n_0$ for some n_0. Hence we can construct N inequivalent regular component sequences at w and so $p\geqslant N$. it follows that $p=\infty$ and (3.3) is also true. (i) is proved

The proof of (ii) is similar as that of (i).

4. SIMPLE BOUNDARY POINTS AND LOCAL CONNECTIVETY

The well- known contiunous extension theorem (see[2])can be stated as the following.

Theorem 4.1 f has a continuous extension to $\overline{D}$ if and only if every boundary point of G is simple.

Furthermore, we have

Theorem 4.2 A compact set E in complex plane is locally connected if and only if every boundary point of its complement is simple and the number of all connected components of E is finite.

Proof If E is locally connected then, of course, the number of all connected components of E is finite. Let E_0 be any component of E, which is also locally connected. By the continuous extension theorem every boundary point of $\overline{C}\setminus E_0$ is simple and then is also a simple boundary point of $\overline{C}\setminus E$. Therefore, all the boundary points of the complement of E are simple.

By Theorem 4.1 the converse is also true because any union of finitely many locally connected sets is also locally connected.

5. SIMPLE BOUNDARY POINTS AND KOEBE ARCS

Let G be a simply connected domain, and g be a meromorphic function in G. If a sequence of Jordan arcs $C_n \subset G$ satisfies

(i) diam $C_n \geqslant \alpha > 0$ and

(ii) $g(z) \to c$ for $z \in C_n, n \to \infty$

for some $c \in \overline{C}$, then it is called a sequence of Koebe arcs with respect to the meromorphic function g. We say that g has no Koebe arcs if no such sequence exists.

Theorem 5.1 If G is bounded then any analytic univalent function in G has no Koebe arcs if and only if every boundary point of G is simple.

Proof (a) If G has a complicated boundary point w then $C(f,\zeta)$ is a continuum for any given $\zeta \in A(w,f)\setminus f^{-1}(w)$. Let $w', w'' \in C(f,\zeta)$ with $|w'-w''| = \text{diam } C(f,\zeta) > 0$. There exist two sequences $\{z_n'\}$ and $\{z_n''\}$ in D tending to ζ with $f(z_n') \to w'$, $f(z_n'') \to w''$ $(n \to \infty)$. We may assume that $z_n' \neq z_n''$, $n = 1, 2, \cdots$. Denote $g = f^{-1}, \gamma_n = [z_n', z_n'']$, $C_n = f(\gamma_n)$, then $\{C_n\}$ is a sequence of Koebe arcs with respect to g which is analytic and univalent in G. It shows that the condition is necessary.

(b) If there exists a sequence $\{C_n\}$ of Koebe arcs with respect to some g which is analytic and univalent in G, then $g(C_n) \to \zeta \in \partial\Omega$ where $\Omega = g(G)$. Let h be a conformal mapping of Ω onto D, then $f = g^{-1}h^{-1}$ is a conformal mapping of D onto G. Note that $A(w,f)$ does not contain any arc by the Riesz uniqueness theorem, there exists a subsequence $\{C_{n_k}\}$ of $\{C_n\}$

such that $h(g(C_{n_k})) \to \zeta_0 \in \partial D$ $(k \to \infty)$. Then f is not continuous at ζ_0 because diam $C_{n_k} \geq \alpha > 0$, $k=1,2,\cdots$. Let $w \in C(f, \zeta_0)$ and $w \neq f(\zeta_0)$ if $f(\zeta_0)$ exists. Therefore, $\zeta_0 \in A(w, f) \setminus f^{-1}(w)$. It shows that w is a complicated boundary point of G and then the condition is sufficient.

6. NOTES

Because of Moore 's plane triode theorem (see [6]) the set of all boundary points of G with pseudo-multiplicity $p \geq 3$ is at most countable.

A Jordan domain has no complicated boundary point. But whether there exists a domain which has no simple boundary point is still an open problem. Now we construct a simply connected domain such that all its simple boundary points is a set of zero linear measure. Let K be the Cantor set in $[0, 1]$, $\eta = \frac{1}{2} + i$, and $E = \bigcup_{x \in K} [x, \eta]$. Then $G = \overline{C} \setminus E$ is such a domain. The set of all its simple boundary points is $K \bigcup \{\eta\}$.

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